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ANALYSIS OF SPECTRAL OPERATORS IN ONE-DIMENSIONAL DOMAINS

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ABSTRACT

We prove results concerning certain projection operators on the space of all polynomials of degree less than or equal to N with respect to a class of one-dimensional weighted Sobolev spaces. These results are useful in the theory of the approximation of partial differential equations with spectral methods.

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I. INTRODUCTION

This paper presents an investigation of a class of projection operators that arises in the analysis of the approximation of differential equations by spectral methods using Chebyshev decomposition.

Some similar operators have been studied before by Canuto-Quarteroni [1] and Maday-Quarteroni [1], but the existing results are not adequate in many applications. In fact they forbid analysis for the error of the approximation by spectral methods of fourth-order problems and, in several instances, second-order problems (see Canuto-Quarteroni [2]).

We first present some background tools required for our analysis. They consist of Sobolev spaces relative to the weight $\omega(x) = (1 - x^2)^{-1/2}$ (this weight arises in the relations of orthogonality of Chebyshev polynomials). We recall and complete results proved by Grisvard [1], [2] concerning interpolation theory between these spaces.

Then we present an analysis of projection operators from these spaces into the set of all polynomials of degree lower than N.

Finally we give an application of the results herein proved to a simple test problem.

We shall give other applications in Maday [1] and shall in a future work extend these results to multidimensional domains. Our aim is to apply such results to the analysis of the approximation of Navier-Stokes equations by spectral methods (see Maday-Metivet [1], [2]).

For some different notions about projection operators that arise in spectal methods, see Tadmor [1].

II. PRELIMINARIES: SOME FUNCTION SPACES

Notations and Basic Properties

Let J be an open interval $]a,b[$ of $\mathbb{R}$ $(a<b)$; we consider a weight function $\rho(x)$, continuous over J, satisfying $\rho(x) \geq \rho_0 > 0$ for any $x \in J$.

Let us set:

$$(2.1) \qquad L^2_\rho(J) = \{\phi : J \to \mathbb{R} \mid \phi \text{ is measurable and } (\phi,\phi)_\rho < +\infty\},$$

equipped with the inner product $(\phi,\psi)_\rho = \int_J \phi(x)\psi(x)\rho(x)dx$. For any integer $s \geq 0$ we set:

$$H^s_\rho(J) = \{\phi \in L^2_\rho(J) \mid \|\phi\|_{s,\rho} < \infty\},$$

where:

$$(2.2) \qquad \|\phi\|_{s,\rho} = ((\phi,\phi))_{s,\rho},$$

this space being equipped with the inner product:

$$(2.3) \qquad ((\phi,\psi))_{s,\rho} = \sum_{k=0}^{s} \left(\frac{d^k\phi}{dx^k}, \frac{d^k\psi}{dx^k}\right)_\rho .$$

Clearly, one has the equality:

$$L^2_\rho(I) = H^0_\rho(I).$$

For any real $s \geq 0$, noninteger, $H^s_\rho(J)$ is defined by interpolation between the space $H^{\bar{s}}_\rho(J)$ and $H^{\bar{s}+1}_\rho(J)$, where $\bar{s}$ represents the integral part of s.

The method of interpolation can be the complex one, the operator's domain one or the trace one (see Lions-Magenes [1] for more details). Besides we define $H^s_{0,\rho}(J)$ as being the closure of $\mathcal{D}(J)$ in $H^s_\rho(J)$. When $\rho = 1$ these spaces are the usual Sobolev spaces denoted by $H^s(J)$ and $H^s_0(J)$ respectively. For the application to spectral methods we are mostly interested in those spaces when $J = I \equiv]-1,+1[$ and $\rho(x) = \omega(x) \equiv \frac{1}{\sqrt{1-x^2}}$. Let us recall some results proved in Grisvard [1], [2] valuable for $J = I, \rho = \omega$ and for $J =]0,1[, \rho = \frac{1}{\sqrt{x}}$.

THEOREM 2.1 (Grisvard [1]):

i) For any real $s > 0$, $s \notin \mathbb{N} + 1/4$ we have:

$$H^s_{0,\rho}(J) = \left[H^{\bar{s}}_{0,\rho}(J),\ H^{\bar{s}+1}_{0,\rho}(J)\right]_{[s-\bar{s}]}. \tag{2.4}$$

ii) For any integral n we have:

$$\left[H^n_{0,\rho}(J),\ H^{n+1}_{0,\rho}(J)\right]_{1/4} \subset H^{n+1/4}_{0,\rho}(J). \tag{2.4'}$$

iii) For any real $s > 0$, $s \notin \mathbb{N} + 1/2$:

$$H^s_\rho(J) \subset C^m(\bar{J}),$$

the space of continuous mapping defined over $\bar{J}$ whose derivative of order $\leq m$ are continuous over $\bar{J}$, with $m = \overline{s - \frac{1}{2}}$.

The trace application defined from $C^\infty(\bar{J})$ into $\mathbb{R}^{2n}$:

$$u \longmapsto \left(u(-1), \frac{du}{dx}(-1), \cdots, \frac{d^n u}{dx^n}(-1), u(1), \frac{du}{dx}(1), \cdots \frac{d^n u}{dx^n}(1)\right)$$

can be extended to a continuous mapping from $H_\rho^{n+1/4+\varepsilon}(I)$ onto $\mathbb{R}^{2n}$ for any $\varepsilon > 0$.

iv) For any real $1/4 < s < 5/4$, $H^s_{0,\rho}(J)$ coincide with the subspace of $H^s_\rho(J)$ of functions vanishing at the real boundaries of J.

v) For any real $s > 1/2$, $H^s_\rho(J)$ is an algebra.

THEOREM 2.2 (Grisvard [2]):

For any $0 \leq q < s < p$, $H^s_\rho(J)$ satisfies the following double topological imbedding:

$$[H^q_\rho(J), H^p_\rho(J)]_{\theta,1} \subset H^s_\rho(J) \subset [H^q_\rho(J), H^p_\rho(J)]_{\theta,\infty}, \tag{2.5}$$

with $\theta = \frac{s-p}{q-p}$, and the notation holds for the real interpolation (see Lions and Peetre [1]).

The two following results can be found in Canuto-Quarteroni [2] and Maday-Quarteroni [1].

THEOREM 2.3:

i) For any real $s \geq 1/4$, $H^s(I) \subset H^{s-1/4}_\omega(I)$.

ii) For any $0 \leq r < s$, the imbedding $H^s_\omega(I) \subset H^r_\omega(I)$ is compact.

In the next section we shall generalize the results (2.4) and (2.4').

Some New Results About Interpolation Between $H^p_{0,\omega}(I)$

This section is devoted to the proof of the following:

THEOREM 2.4: For any $0 \leq q < s < p$ not in $\mathbb{N} + 1/4$ we have:

$$[H^q_{0,\omega}(I), H^p_{0,\omega}(I)]_{[\frac{s-p}{q-p}]} = H^s_{0,\omega}(I).$$

This theorem is a consequence of the two following lemma:

LEMMA 2.1: For any integer $p \leq n$, we have:

$$u \in H^n_{0,\omega}(I) \quad \Rightarrow \quad \frac{d^p u}{dx^p} \in L^2_{\omega^{4(n-p)+1}}(I).$$

PROOF: It is an easy matter to check that this result is a consequence of

$$\text{(2.6)} \qquad u \in H^n_{0,\frac{1}{\sqrt{x}}}(0,1) \quad \Rightarrow \quad \frac{d^p u}{dx^p} \in L^2_{(\frac{1}{\sqrt{x}})^{4(n-p)+1}}(0,1),$$

(we shift the difficulties at ±1 onto 0). So let u be in $H^n_{0,\frac{1}{\sqrt{x}}}(0,1)$; from Theorem 2.1 (point iv) we have, for any $0 \leq p < n$:

$$\frac{d^p u}{dx^p}(0) = 0,$$

hence

$$\int_0^x \frac{d^{p+1} u}{dx^{p+1}}(t)dt = \frac{d^p u}{dx^p}(x).$$

Besides, from Lemma 6.2.1 of Nečas [1] we have, for any $\alpha < 1$ and any v such that $\int_0^1 v^2(x)x^\alpha \, dx < \infty$:

$$\text{(2.7)} \qquad \int_0^1 \left(\int_0^x |v(x)|\right)^2 x^{\alpha-2} \, dx \leq \left(\frac{1}{1-\alpha}\right) \int_0^1 |v(x)|^2 x^\alpha \, dx,$$

taking then $\alpha = -\frac{1}{2} - 2\big(n - (p+1)\big)$ and $v = \dfrac{d^{p+1}u}{dx^{p+1}}$ we obtain:

$$\int_0^1 \left(\frac{d^p u}{dx^p}\right)^2 x^{-1/2 - 2(n-p)} \, dx \leq C \int_0^1 \left(\frac{d^{p+1} u}{dx^{p+1}}\right)^2 x^{-1/2 - 2(n-(p+1))} \, dx,$$

and (2.6) holds by induction over p.

LEMMA 2.2: For any integer $n > 0$, the mapping $u \longmapsto u\omega^{1/2}$ is an homeomorphism from $H^n_{0,\omega}(I)$ onto $H^n_0(I)$.

<u>PROOF:</u> Here again we prove the result for the weight $\dfrac{1}{\sqrt{x}}$, say:

$$\text{(2.8)} \qquad u \longmapsto ux^{-1/4} \text{ is an homeomorphism from } H^n_{0,\frac{1}{\sqrt{x}}}(0,1) \text{ onto } H^n_0(0,1).$$

Let $\phi \in \mathcal{D}(0,1)$, then, for $0 \leq m \leq n$:

$$\frac{d^m}{dx^m}(\phi x^{-1/4}) = \sum_{p=0}^{m} C^p_m \frac{d^p \phi}{dx^p} \frac{d^{m-p}(x^{-1/4})}{dx^{m-p}}$$

$$= \sum_{p=0}^{m} C^p_m D^p_m \frac{d^p \phi}{dx^p} x^{-1/4 - (m-p)},$$

with $D^p_m = \left[-1/4 - (m-p+1)\right] D^{p+1}_m$ and $D^m_m = 1$. From Lemma 2.1 we then get:

$$\left\|\frac{d^m}{dx^m}(\phi x^{-1/4})\right\|_{0,1} \leq C\|\phi\|_{m,1/\sqrt{x}} \leq C\|\phi\|_{n,1/\sqrt{x}} \ ;$$

summing up these estimates for $0 \leq m \leq n$ we derive:

$$\|\phi x^{-1/4}\|_{n,1} \leq C\|\phi\|_{n,1/\sqrt{x}} \ . \tag{2.9}$$

Inversely, let us prove that, for any $\phi \in \mathcal{D}(I)$:

$$\|\phi x^{1/4}\|_{n,1/\sqrt{x}} \leq C\|\phi\|_{n,1}. \tag{2.10}$$

From Hardy's inequality (Lemma 2.5.1 of Nečas [1]) we derive by induction that, for any $0 \leq p \leq m \leq n$:

$$\left\|\frac{d^p\phi}{dx^p}\right\|_{0,x^{-2(m-p)}} \leq C\|\phi\|_{m,1}, \tag{2.11}$$

besides:

$$\frac{d^m(\phi x^{1/4})}{dx^m} = \sum_{p=0}^{m} C_m^p \frac{d^p\phi}{dx^p} \frac{d^{m-p}}{dx^{m-p}}(x^{1/4})$$

$$= \sum_{p=0}^{m} C_m^p D_m'^p \frac{d^p\phi}{dx^p} x^{1/4-(m-p)},$$

with:

$$D_m'^p = [\,1/4 - (m-p+1)]D_m'^{p+1} \quad \text{and} \quad D_m'^m = 1. \cdot$$

Then using (2.11) we get:

$$\frac{d^m(\phi x^{1/4})}{dx^m} \in L^2_{1/\sqrt{x}}(0,1),$$

and (2.10) is derived by summing up these results for $0 \leq m \leq n$. We can now achieve (2.8) as a consequence of (2.9) and (2.10).

We can now prove the main result of this section.

PROOF OF THEOREM 2.4: From (2.4) and Lemma 2.2 we deduce that the mapping $u \longmapsto u\omega^{1/2}$ is an homeomorphism from $H^s_{0,\omega}(I)$ onto $H^s_0(I)$ for any $s \geq 0$ not in $\mathbb{N} + 1/4 \cap \mathbb{N} + 1/2$ (see Lions-Magenes [1] for more details about the properties of spaces of interpolation).

Let us recall that, for any $q \leq s \leq p$ not in $\mathbb{N} + 1/2$ we have (see Lions-Magenes [1]):

(2.12)
$$H^s_0(I) = [H^q_0(I),\ H^p_0(I)]_{\frac{s-q}{p-q}} .$$

From the previous homeomorphism we deduce that, for any $q \leq s \leq p$ not in $\{\mathbb{N} + 1/2\} \cup \{\mathbb{N} + 1/4\}$:

(2.13)
$$H^s_{0,\omega}(I) = [H^q_{0,\omega}(I),\ H^p_{0,\omega}(I)]_{\frac{s-q}{p-q}} .$$

Let us remark now that the values of p, q, s in $\mathbb{N} + 1/2$ have only been excluded due to (2.12), these values can now be recovered thanks to the reiteration theorem (Theorem I.6.1 of Lions-Magenes [1]).

III. APPROXIMATION RESULTS OF PROJECTION OPERATOR IN WEIGHTED SOBOLEV SPACES

The previous theorem leads us to define over $H^r_{0,\omega}(I)$ a new scalar product. Indeed, for p not in $\mathbb{N}+1/4 \cup \mathbb{N}+1/8$, $H^p_{0,\omega}(I)$ can be seen as the interpolate $1/2$ between $L^2_\omega(I)$ and $H^{2p}_{0,\omega}(I)$ and for p in $\mathbb{N}+1/8$, $H^p_{0,\omega}(I)$ can be seen as the interpolate $1/3$ between $L^2_\omega(I)$ and $H^{3p}_{0,\omega}(I)$.

If we consider the domain operator interpolation, this find expression in the existence of a selfadjoint operator Λ_r such that:

* if $r \in \mathbb{N}+1/8$, the domain $D(\Lambda^3_r)$ of the operator Λ^3_r in $L^2_\omega(I)$ is $H^{3r}_{0,\omega}(I)$ if $r \notin \mathbb{N}+1/8$, the domain of $D(\Lambda^2_r)$ of the operator Λ^2_r in $L^2_\omega(I)$ is $H^{2r}_{0,\omega}(I)$.
* The domain $D(\Lambda_r)$ of the operator Λ_r in $L^2_\omega(I)$ is $H^r_{0,\omega}(I)$ if $r \notin \mathbb{N}+1/4$ and is included in $H^r_{0,\omega}(I)$ if $r \in \mathbb{N}+1/4$.

Moreover:

$$(u,v) \longmapsto (((u,v)))_{r,\omega} \equiv (\Lambda_r u, \Lambda_r v)_\omega, \tag{3.1}$$

is a scalar product whose associated norm is equivalent to the one defined in (2.2) if $r \notin \mathbb{N}+1/4$.

Let us define now $P_{r,N}$ as the projection operator from $H^r_{0,\omega}(I)$ over S^r_N with respect to the previous scalar product with:

$$S^r_N = S_N \cap H^r_{0,\omega}(I),$$

$$S_N = \{\phi \text{ defined over } I \mid \phi \text{ is a polynomial of degree } \leq N\}.$$

LEMMA 3.1: Let $0 \leq \nu \leq r \leq \sigma$ with $\sigma \notin \mathbb{N} + 1/4$ we have, for any $\phi \in H^{\sigma}_{\omega}(I) \cap H^{r}_{0,\omega}(I)$:

$$\|\phi - P_{r,N}\, \phi\|_{\nu,\omega} \leq C N^{\nu-\sigma}\, \|\phi\|_{\sigma,\omega}. \tag{3.2}$$

REMARK 3.1: The case $\nu = r = 0$ has been studied in Canuto-Quarteroni [1], the case $0 \leq \nu \leq r = 1$ has been looked at in Maday-Quarteroni [1] (note that the dependence of the constant is then $C(\sigma) = C\cdot(\sigma!)$). Moreover it is proved that no optimal bound was possible for $H^{\nu}_{0,\omega}(I)$ norms with $\nu > r$. Indeed, for example:

$$\|\phi - P_{0,N}\, \phi\|_{\nu,\omega} < C N^{2\nu-\sigma}\, \|\phi\|_{\sigma,\omega}. \tag{3.4}$$

It is often necessary (see Canuto-Quarteroni [2], Maday-Metivet [2], Maday [1], and (4.14)) to obtain optimal results in higher norms.

PROOF: We shall only consider the case $r \notin \mathbb{N} + 1/8$ for simplicity. The proof is divided in two stages

i) We first prove (3.2) by induction over r in $\mathbb{N}$. So, let us assume that (3.2) is true for $s < r$ in $\mathbb{N}$; let $\phi \in H^{r}_{0,\omega}(I)$; then $\phi_x \in H^{r-1}_{0,\omega}(I)$ and $P_{r-1,N-1}(\phi_x) \in S^{r-1}_{N-1}$. Moreover if $\phi(-1) = \phi(1) = 0$ we have:

$$\alpha \equiv \int_{-1}^{1} P_{r-1,N-1}(\phi_x)(t)dt = \int_{-1}^{1} [P_{r-1,N-1}(\phi_x) - \phi_x](t)dt.$$

From the Cauchy-Schwarz inequality we derive:

$$|\alpha| \le \left(\int_{-1}^{1} (P_{r-1,N-1}(\phi_x) - \phi_x)^2(t)\omega(t)dt\right)^{1/2} \left(\int_{-1}^{1} (\omega(t))^{-1}\, dt\right)^{1/2}$$

$$\le C\|P_{r-1,N-1}(\phi_x) - \phi_x\|_{0,\omega};$$

hence, from the induction hypothesis:

$$|\alpha| \le CN^{1-\sigma}\, \|\phi_x\|_{\sigma-1,\omega}. \tag{3.5}$$

Finally we have:

$$R_N(x) = \int_{-1}^{x} \left[P_{r-1,N-1}(\phi_x)(t) - \frac{\alpha(1 - t^2)^{r-1}}{\int_{-1}^{1} (1 - x^2)^{r-1}\, dx}\right] dt \in S_N^r.$$

Due to the Poincaré-like inequality, the polynomial satisfies the following:

$$\|\phi - R_N\|_{r,\omega} \le \|(\phi - R_N)_x\|_{r-1,\omega},$$

the induction hypothesis, and (3.5) gives us:

$$\|\phi - R_N\|_{r,\omega} \le C\left(N^{(r-1)-(\sigma-1)} + N^{(1-\sigma)}\right) \|\phi_x\|_{\sigma-1,\omega}$$

$$\le CN^{r-\sigma}\, \|\phi\|_{\sigma,\omega}.$$

From the equivalence of the norms $\|\ \|_{r,\omega}$ and $|||\cdot|||_{r,\omega}$, and the identity:

$$|||\phi - P_{r,N}\,\phi|||_{r,\omega} = \inf_{\phi_N \in S_N^r} |||\phi - \phi_N|||_{r,\omega},$$

we obtain for any ϕ in $H^{\sigma}_{\omega}(I) \cap H^{r}_{0,\omega}(I)$:

$$|||\phi - P_{r,N}\,\phi|||_{r,\omega} \leq CN^{r-\sigma}\,\|\phi\|_{\sigma,\omega}. \tag{3.6}$$

Besides, since the operator Λ_r is selfadjoint, we have:

$$\|\phi - P_{r,N}\,\phi\|_{0,\omega} = \inf_{\psi \in L^2_{\omega}(I)} \frac{(\phi - P_{r,N}\,\phi, \psi)_{\omega}}{\|\psi\|_{0,\omega}}$$

$$= \inf_{\psi \in L^2_{\omega}(I)} \frac{(\Lambda_r(\phi - P_{r,N}\,\phi),\ \Lambda_r^{-1}\,\psi)_{\omega}}{\|\psi\|_{0,\omega}} .$$

From (3.1) we then get:

$$\|\phi - P_{r,N}\,\phi\|_{0,\omega} = \inf_{\psi \in L^2_{\omega}(I)} \frac{(((\phi - P_{r,N}\,\phi,\ \Lambda_r^{-2}\,\psi)))_{r,\omega}}{\|\psi\|_{0,\omega}} . \tag{3.7}$$

By definition of $P_{r,N}$ we have, for any ψ in $L^2_{\omega}(I)$:

$$(((\phi - P_{r,N}\,\phi,\ P_{r,N}(\Lambda_r^{-2}\,\psi))))_{r,\omega} = 0;$$

hence

$$\|\phi - P_{r,N}\,\phi\|_{0,\omega} = \inf_{\psi \in L^2_{\omega}(I)} \frac{(((\phi - P_{r,N}\,\phi, (\Lambda_r^{-2}\,\psi) - P_{r,N}(\Lambda_r^{-2}\,\psi))))_{r,\omega}}{\|\psi\|_{0,\omega}}$$

$$\leq |||\phi - P_{r,N}\,\phi|||_{r,\omega} \inf_{\psi \in L^2_{\omega}(I)} \frac{|||(\Lambda_r^{-2}\,\psi) - P_{r,N}(\Lambda_r^{-2}\,\psi)|||_{r,\omega}}{\|\psi\|_{0,\omega}} .$$

Due to (3.6) we then derive:

$$\|\phi - P_{r,N}\,\phi\|_{0,\omega} \le CN^{r-\sigma}\,\|\phi\|_{\sigma\,\omega}\, N^{-r} \inf_{\psi \in L^2_\omega(I)} \frac{\|\Lambda_r^{-2}\,\psi\|_{2r,\omega}}{\|\psi\|_{0,\omega}}$$

$$\le CN^{-\sigma}\,\|\phi\|_{\sigma\,\omega} \inf_{\psi \in L^2_\omega(I)} \frac{\|\Lambda_r^2(\Lambda_r^{-2}\,\psi)\|_{0,\omega}}{\|\psi\|_{0,\omega}}$$

$$\le CN^{-\sigma}\,\|\phi\|_{\sigma,\omega}.$$

Now, from the two estimates, valuable for any $\phi \in H^\sigma_\omega(I) \cap H^r_{0,\omega}(I)$:

$$\|\phi - P_{r,N}\,\phi\|_{r,\omega} \le CN^{r-\sigma}\,\|\phi\|_{\sigma\,\omega}$$

$$\|\phi - P_{r,N}\,\phi\|_{0,\omega} \le CN^{-\sigma}\,\|\phi\|_{\sigma,\omega}$$

we derive that for any $\theta \in]0,1[$:

$$\|\phi - P_{r,N}\,\phi\|_{[L^2_\omega(I),H^r_{0,\omega}(I)]_\theta} \le CN^{\theta r-\sigma}\,\|\phi\|_{\sigma,\omega}.$$

Due to (2.4) and (2.4´) we deduce that, for any $0 \le \nu \le r$:

$$\|\phi - P_{r,N}\,\phi\|_{\nu,\omega} \le CN^{\nu-\sigma}\,\|\phi\|_{\sigma,\omega}. \tag{3.8}$$

ii) Let us now prove (3.3) for nonintegral values of r. Let $\phi \in \mathcal{D}(I)$, from step (i) we know that, for any $\sigma \ge \overline{r+1}$, $\sigma \notin \mathbb{N} + 1/4$

$$\inf_{\phi_N \in S_N^{\overline{r+1}}} \|\phi - \phi_N\|_{\overline{r+1}\ \omega} \leq CN^{\overline{r+1}-\sigma} \|\phi\|_{\sigma\ \omega},$$

$$\inf_{\phi_N \in S_N^{\overline{r+1}}} \|\phi - \phi_N\|_{\overline{r},\omega} \leq CN^{\overline{r}-\sigma} \|\phi\|_{\sigma,\omega}$$

(see (3.8) with $\nu = \overline{r+1}$ and $\nu = \overline{r}$ respectively). These two estimates are equivalent to the following one:

$$\|\dot{\phi}\|_{H_{0,\omega}^{\overline{r+1}}(I)/S_N^{\overline{r+1}}} \leq CN^{\overline{r+1}-\sigma} \|\phi\|_{\sigma,\omega},$$

$$\|\dot{\phi}\|_{H_{0,\omega}^{\overline{r}}(I)/S_N^{\overline{r+1}}} \leq CN^{\overline{r}-\sigma} \|\phi\|_{\sigma,\omega}.$$

Due to the interpolation of quotient spaces (see Lions-Magenes [1] Lemma I.13.2) we have, for any $\theta \in]0,1[$:

$$\|\dot{\phi}\|_{[H_{0,\omega}^{\overline{r}}(I),H_{0,\omega}^{\overline{r+1}}(I)]_\theta/S_N^{\overline{r+1}}} \leq CN^{\overline{r}+\theta-\sigma} \|\phi\|_{\sigma,\omega}.$$

From (2.4), (2.4′) we deduce (we take $\theta = r - \overline{r}$):

$$\|\dot{\phi}\|_{H_{0,\omega}^{r}(I)/S_N^{\overline{r+1}}} \leq CN^{r-\sigma} \|\phi\|_{\sigma,\omega},$$

so that, for any $\sigma \geq \overline{r+1}$ $\sigma \notin \mathbb{N} + 1/4$:

(3.9) $$\|\phi - P_{r,N}\,\phi\|_{r,\omega} < CN^{r-\sigma}\,\|\phi\|_{\sigma\,\omega}.$$

By definition we know that, for any $r \notin \mathbb{N} + 1/4$.

$$\|\phi - P_{r,N}\|_{r,\omega} \leq C\|\phi\|_{r,\omega};$$

hence, (3.9) holds for any $\sigma \geq r$, $\sigma \notin \mathbb{N} + 1/4$ and any $r \in \mathbb{R}^{+*}$. The estimates in lower order norms are obtained following the same lines as in i).

We are now interested in the approximation of the spaces $H^r_\omega(I) \cap H^s_{0,\omega}(I)$ for $s \in \mathbb{N}$, $0 \leq s \leq r$ by polynomials therein contained.

For simplicity of expositure we shall consider the case $s = 0$. From point iii) of Theorem 2.4 we can easily exhibit for any $\phi \in H^r_\omega(I)$ a polynomial ϕ_0 of degree $\leq 2r - 1$ such that: $\phi - \phi_0 \in H^r_{0,\omega}(I)$ and for any real p:

(3.10) $$\|\phi_0\|_{p,\omega} \leq C\|\phi\|_{r,\omega}.$$

Due to the previous lemma, we have, for any $0 \leq \nu \leq r \leq \sigma \notin \mathbb{N} + 1/4$:

$$\|(\phi - \phi_0) - (P_{r,N}(\phi - \phi_0))\|_{\nu,\omega} \leq CN^{\nu-\sigma}\,\|\phi - \phi_0\|_{\sigma\,\omega},$$

and (3.10) then implies:

$$\|\phi - (\phi_0 + P_{r,N}(\phi - \phi_0))\|_{\nu,\omega} \leq CN^{\nu-\sigma}\,\|\phi\|_{\sigma\,\omega}.$$

For N large enough (more precisely $N \geq 2r - 1$) we then get the existence of an operator $\tilde{P}$ from $H^r_\omega(I)$ onto S_N such that:

$$\|\phi - \tilde{P}(\phi)\|_{\nu,\omega} \leq CN^{\nu-\sigma} \|\phi\|_{\sigma,\omega}.$$

This estimate provides an answer to our question in the case $s = 0$. The same proof can be done to build an operator from $H^r_\omega(I)$ $H^s_{0,\omega}(I)$ onto S^s_N satisfying analogous bounds. This leads us to state the main theorem of this paper:

THEOREM 3.1: Let $(\nu,r,\sigma) \in \mathbb{R}$, $\sigma \notin \mathbb{N} + 1/4$, $s \in \mathbb{N}$, $0 \leq s \leq r$, $0 \leq \nu \leq r \leq \sigma$. There exists an operator $\Pi^{s,0}_{r,N}$ from $H^r_\omega(I) \cap H^s_{0,\omega}(I)$ onto S^s_N such that, for any $\phi \in H^\sigma_\omega(I) \cap H^s_{0,\omega}(I)$ we have:

$$\|\phi - \Pi^{s,0}_{r,N}\,\phi\|_{\nu,\omega} \leq CN^{\nu-\sigma} \|\phi\|_{\sigma,\omega}.$$

IV. AN APPLICATION

Definition of the Problem

In order to explain how the previous results can be applied, we shall study an approximation of the very simple problem:

Find ψ defined over I such that:

$$(4.1)\qquad \begin{cases} \dfrac{d^4\psi}{dx^4} = f & \text{over } I, \\ \\ \psi = \dfrac{d\psi}{dx} = 0 & \text{at } \pm 1. \end{cases}$$

(This problem provides a first step for the analysis of Stokes and Navier-Stokes problems in the ψ-formulation; see Maday-Metivet [1], [2].) Let us define $\mathcal{H}_\omega^{-2}(I)$ as follows:

$$\mathcal{H}_\omega^{-2}(I) = \{f \in \mathcal{D}'(I) | \exists g \in L_\omega^2(I): \quad f = \frac{d^2 g}{dx^2}\} .$$

We now want to prove the following:

THEOREM 4.1: Let $f \in \mathcal{H}_\omega^{-2}(I)$; then there exists one and only one solution ψ to the problem (4.1) in the space $H_{0,\omega}^2(I)$.

This theorem is a very simple consequence of Lax Milgram lemma and the two following lemmas.

LEMMA 4.1: There exist two positive constants δ_1 and δ_2 such that, for any ϕ in $H_{0,\omega}^2(I)$:

$$\int_I \phi^2 \omega^9 \leq \delta_1 \int_I \frac{d\phi}{dx}^2 \omega^5 \leq \delta_2 \int_I \left(\frac{d^2\phi}{dx^2}\right)^2 \omega.$$

This lemma is a corollary of Lemma 2.1.

LEMMA 4.2: There exist 3 positive constants $\alpha,\beta\ \gamma$ such that for any $(\phi,\psi) \in H^2_\omega(I)xH^2_{0,\omega}(I)$:

$$(\psi_{xx},(\psi\omega)_{xx}) \geq \alpha\|\psi\|^2_{2,\omega}, \tag{4.2}$$

$$\|\psi\|_{2,\omega} \leq \beta\|\psi_{xx}\|_{0,\omega}, \tag{4.3}$$

$$(\phi_{xx},(\psi\omega)_{xx})_\omega \leq \gamma\|\phi_{xx}\|_{0,\omega}\ \|\psi_{xx}\|_{0,\omega}. \tag{4.4}$$

<u>PROOF</u>:

i) We first note that (4.3) is an easy consequence of the previous lemma, and is an equivalent to the Poincaré inequality.

ii) Next, we get the following equalities, for any $\psi \in \mathcal{D}(I)$:

$$\int_I \psi_{xx}(\psi\omega)_{xx}\ dx = \int_I \psi^2_{xx}\ \omega dx + 2\int_I \psi_{xx}\ \psi_x\ \omega_x\ dx + \int_I \psi_{xx}\ \psi\omega_{xx}\ dx$$

$$= \int_I \psi^2_{xx}\ \omega dx + \int_I (\psi^2_x)_x\ \omega_x\ dx + \int_I (\psi_x)_x\ \psi\omega_{xx}\ dx$$

$$= \int_I \psi^2_{xx}\ \omega dx - \int_I \psi^2_x\ \omega_{xx}\ dx - \int_I \psi_x(\psi\omega_{xx})_x\ dx$$

$$= \int_I \psi^2_{xx}\ \omega dx - 2\int_I \psi^2_x\ \omega_{xx}\ dx + 1/2\int_I \psi^2\ \omega_{xxxx}\ dx$$

let us note that:

$$\omega_{xx} = (1 + 2x^2)\omega^5,$$

$$\omega_{xxxx} = (9 + 72x^2 + 24x^4)\omega^9;$$

hence:

(4.5)

$$\int_I \psi_{xx}(\psi\omega)_{xx}\, dx = \int_I \psi_{xx}^2\, dx - 2\int_I \psi_x^2(1 + 2x^2)\omega^5\, dx + 1/2 \int_I \psi^2(9 + 72x^2 + 24x^4)\omega^9\, dx.$$

Besides, let us set:

$$P \equiv \int_I \left(\psi_{xx}\,\omega + 2x\psi_x\,\omega^3 + (2x^2 + 10^{-2})\psi\omega^5\right)^2 \omega^{-1} dx;$$

P is ≥ 0 and, an easy calculation gives:

$$P \equiv \int_I \psi_{xx}(\psi\omega)_{xx} - 2.10^{-2} \int_I \psi_x^2\,\omega^5 - \int_I (5{,}78x^2 + 0{,}4839)\psi^2\,\omega^9,$$

so that

$$\int_I \psi_x^2\,\omega^5 \leq 50\int_I \psi_{xx}(\psi\omega)_{xx},$$

from which we derive:

$$\int_I \psi_x^2(1 + 2x^2)\omega^5 \leq 150\int_I \psi_{xx}(\psi\omega)_{xx}.$$

Using that inequality in (4.5) we obtain $\alpha > 0$ such that:

$$\int_I \psi_{xx}(\psi\omega)_{xx} \geq \alpha\beta\int_I \psi_{xx}^2\ \omega,$$

and (4.2) is an consequence of (4.3).

iii) Finally, let us note that for any (ϕ,ψ) in $H_\omega^2(I)xH_{0,\omega}^2(I)$ we have:

$$(4.6)\qquad (\phi_{xx},(\psi\omega)_{xx})_1 = (\phi_{xx},\psi_{xx})_\omega + 2\int_I \phi_{xx}\ \psi_x\ \omega_x + \int_I \phi_{xx}\ \psi\omega_{xx}.$$

The following inequality is simple:

$$(4.7)\qquad |(\phi_{xx},\psi_{xx})_\omega| \leq \|\phi_{xx}\|_{0,\omega}\ \|\psi_{xx}\|_{0,\omega}.$$

Let us examine the second term:

$$|\int_I \phi_{xx}\ \psi_x\ \omega_x| = |\int_I \phi_{xx}(\psi_x\ \omega_x\ \omega^{-1})\omega| \leq |\int_I \phi_{xx}^2\ \omega|^{1/2}\,|\int_I \psi_x^2\ \omega_x^2\ \omega^{-1}|^{1/2};$$

since $\omega_x^2\ \omega^{-1} = x^2\ \omega^5$, we derive from Lemma 4.1 that:

$$(4.8)\qquad |\int_I \phi_{xx}\ \psi_x\ \omega_x| \leq C|\phi_{xx}|_{0,\omega}\ |\psi_{xx}|_{0,\omega}.$$

We obtain, in a similar way:

$$(4.9)\qquad |\int_I \phi_{xx}\ \psi\omega_{xx}| \leq C|\phi_{xx}|_{0,\omega}\ |\psi_{xx}|_{0,\omega},$$

so that (4.4) is a consequence of (4.6)-(4.9).

PROOF OF THEOREM 4.1: Let $f = g_{xx}$ be in $H_{\omega}^{-2}(I)$. Problem (4.1) is equivalent to the following:

$$(4.10)\quad \begin{cases} \text{Find } \psi \text{ in } V = H_{0,\omega}^{2}(I) \text{ such that, for any } \phi \text{ in } V: \\ \\ \int_I \psi_{xx}(\phi\omega)_{xx} = \int_I g(\phi\omega)_{xx}. \end{cases}$$

The bilinear form a defined by: For any (χ,ϕ) in V^2:

$$(4.11)\qquad a(\phi,\chi) = \int_I \phi_{xx}(\chi\omega)_{xx},$$

is continuous and elliptic over V (see Lemma 4.2), and Lax-Milgram lemma gives the existence and uniqueness of a solution of (4.10) hence of (4.1).

Approximation of Problem 4.1

We are interested in approximating the solution of (4.1) by a polynomial of degree $\leq N$. We use a Galerkin method approach known as Spectral Method (see Gottlieb-Orszag [1] for more details); hence from (4.10) we derive an approximate problem:

$$(4.12)\quad \begin{cases} \text{Find } \psi_N \text{ in } V_N = S_N^2 \text{ such that, for any } \phi \text{ in } V_N: \\ \\ \int_I \psi_{Nxx}(\phi\omega)_{xx} = \int_I g(\phi\omega)_{xx}. \end{cases}$$

From Lemma 4.2 we know that problem (4.12) is wellposed in the sense that there exists one and only one solution. Moreover, we derive from (4.10) and (4.12):

$$a(\psi - \psi_N, \phi) = 0 \qquad \text{for any } \phi \text{ in } V_N,$$

so that (remind $\Pi^{2,0}_{2,N}\, \psi \in V_N$):

$$a(\psi - \psi_N, \psi - \psi_N) = a\big(\psi - \psi_N, \phi - \Pi^{2,0}_{2,N}\, \psi\big). \tag{4.13}$$

Due to Theorem 3.1 and Lemma 4.2, we then obtain the following

THEOREM 4.2: There exists one and only one solution ψ_N to problem (4.12); moreover it verifies, as soon as $\psi \in H^{\sigma}_{\omega}(I) \cap H^{2}_{0,\omega}(I)$:

$$\psi - \psi_N{}_{2,\omega} \leq CN^{2-\sigma}\ \psi_{\sigma,\omega}. \tag{4.14}$$

REMARK 4.1: The previous estimate is an optimal one in the sense that no polynomial of S^2_N is asymptotically nearer from the solution ψ than the solution of the approximate problem.

REMARK 4.2: The previous theorem will be extended in a future paper where we shall consider a pseudospectral method (much more efficient from a computational point of view) for approximating a one-dimensional fourth order equation.

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16. Abstract

We prove results concerning certain projection operators on the space of all polynomials of degree less than or equal to N with respect to a class of one-dimensional weighted Sobolev spaces. These results are useful in the theory of the approximation of partial differential equations with spectral methods.

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